# Year 13 Mathematics

# EAS 3.6

## Differentiation

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## Contents



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# Differentiation 3.6

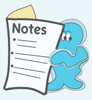
This achievement standard involves applying differentiation methods in solving problems.

| Achievement | Achievement with Merit | Achievement with Excellence |
|---|---|---|
| • Apply differentiation methods in solving problems. | • Apply differentiation methods, using relational thinking, in solving problems. | • Apply differentiation methods, using extended abstract thinking, in solving problems. |

- ◆ This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives
  - ❖ Identify discontinuities and limits of functions.
  - ❖ Choose and apply a variety of differentiation techniques to functions and relations using analytical methods.
- ◆ Apply differentiation methods in solving problems involves:
  - ❖ selecting and using methods
  - ❖ demonstrating knowledge of concepts and terms
  - ❖ communicating using representations.
- ◆ Relational thinking involves one or more of:
  - ❖ selecting and carrying out a logical sequence of steps
  - ❖ connecting different concepts or representations
  - ❖ demonstrating understanding of concepts
  - ❖ forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.
- ◆ Extended abstract thinking involves one or more of:
  - ❖ devising a strategy to investigate or solve a problem
  - ❖ identifying relevant concepts in context
  - ❖ developing a chain of logical reasoning, or proof
  - ❖ forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.
- ◆ Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.
- ◆ Methods include a selection from those related to:
  - ❖ derivatives of power, exponential, and logarithmic (base e only) functions
  - ❖ derivatives of trigonometric (including reciprocal) functions
  - ❖ optimisation
  - ❖ equations of normals
  - ❖ maxima and minima and points of inflection
  - ❖ related rates of change
  - ❖ derivatives of parametric functions
  - ❖ chain, product, and quotient rules
  - ❖ equations of normals
  - ❖ properties of graphs (limits, differentiability, continuity, concavity).



## Limits from Graphs

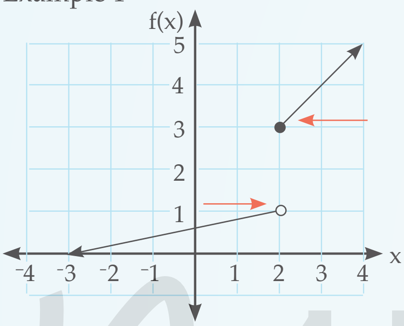
A limit exists if when approached from the left or the right hand side the limit is finite and the same.

If f(x) gets closer and closer to a specific value L as x approaches a chosen value 'a' from the right, then we say that the limit of f(x) as x approaches 'a' from the right is L.

If f(x) gets closer and closer to a specific value L as x approaches a chosen value 'a' from the left, then we say that the limit of f(x) as x approaches 'a' from the left is L.

If the limit of f(x) as x approaches 'a' is the same from both the right and the left, then we say that the limit of f(x) as x approaches 'a' is L.

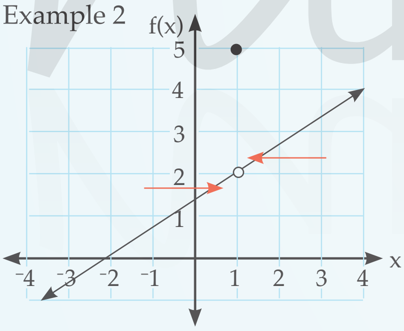
Example 1




As $x \to 2$ from the left, $f(x) \to 1$ and as $x \to 2$ from the right, $f(x) \to 3$.

Therefore $\lim\limits_{x \to 2} f(x)$ does not exist because the left limit ≠ the right limit.

Example 2

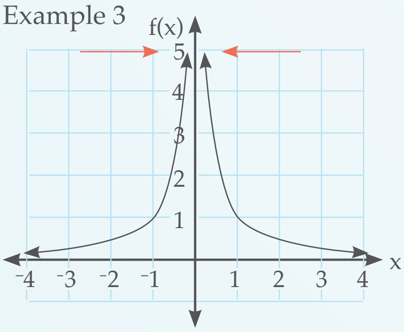



As $x \to 1$ from the left, $f(x) \to 2$ and as $x \to 1$ from the right, $f(x) \to 2$.

Therefore $\lim\limits_{x \to 1} f(x)$ exists because the left limit = the right limit.

So $\lim\limits_{x \to 1} f(x) = 2$, but $f(1) = 5$.

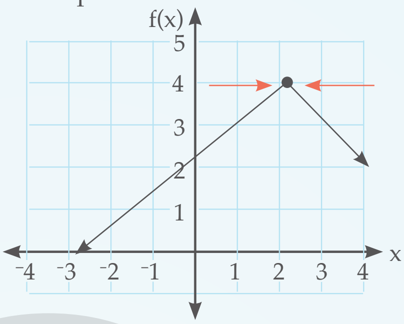
Example 3




As $x \to 0$ from the left, f(x) gets larger and larger $(+\infty)$ and as $x \to 0$ from the right, f(x) gets larger and larger $(+\infty)$.

Therefore $\lim\limits_{x \to 0} f(x)$ does not exist because $(+\infty)$ is not a finite limit.

Example 4

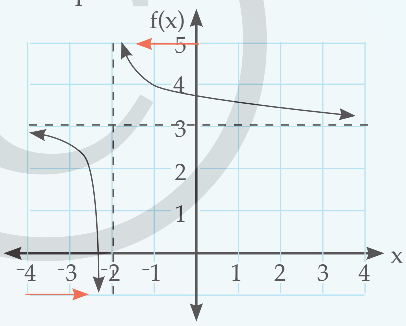



As $x \to 2$ from the left, $f(x) \to 4$ and as $x \to 2$ from the right, $f(x) \to 4$.

Therefore $\lim\limits_{x \to 2} f(x) = 4$, because the left limit = the right limit.

Also $f(2) = 4$.

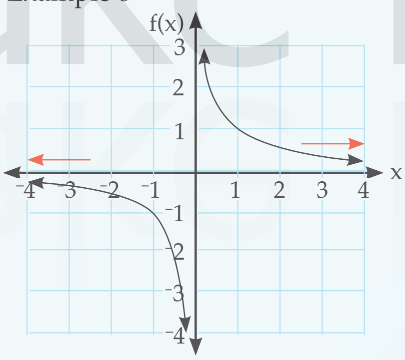
Example 5




As $x \to {}^{-}2$ from the left, f(x) gets smaller and smaller $({}^{-}\infty)$ and as $x \to {}^{-}2$ from the right, f(x) gets larger and larger $(+\infty)$.

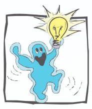
Therefore $\lim\limits_{x \to {}^{-}2} f(x)$ does not exist because $({}^{-}\infty)$ and $(+\infty)$ are not finite limits and the left limit ≠ the right limit.

Example 6




As $x \to \infty$ $f(x) \to 0$ and as $x \to {}^{-}\infty$ $f(x) \to 0$.

Therefore $\lim\limits_{x \to {}^{-}\infty} f(x) = 0$ and $\lim\limits_{x \to \infty} f(x) = 0$.



**The use of the infinity symbol is shorthand for saying the limit does not exist.**

**Infinity is not a number, and an equation or function can never be equal to infinity.**

**We can say that a limit is equal to infinity to indicate that the function as x approaches 'a' gets larger and larger indefinitely, but we can never say that f(x) itself is equal to infinity, because no function can ever reach a value of infinity at every single point on the graph of f(x).**

# Differentiation of Products of Two or More Functions

## Differentiating Products

In this section we are concerned with differentiating the product of two functions, i.e. one function multiplied by the other such as $f(x) = (2x + 3)(x - 4)$.

In some instances we could multiply out the two functions first and then differentiate the result, but in most situations this is not a practical option.

## The Product Rule

For any function which is expressed as a product of two functions

$$h(x) = f(x)g(x)$$

then

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

The product rule using different notation is:

$$y = uv$$

$$y' = u'v + uv'$$

or $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

**Differentiating each function first then multiplying these derivatives, will NOT produce the correct answer.**

**At Achievement and Merit level you only need to be able to use this formula. For Excellence you may be required to understand the proof.**

## Proof of the Differentiation of Products

If $k(x) = f(x)g(x)$ then $k'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

Note: $f'(x) = \frac{f(x+h) - f(x)}{h}$, and therefore $f(x + h) = hf'(x) + f(x)$ by cross-multiplying

and $g'(x) = \frac{g(x+h) - g(x)}{h}$, and therefore $g(x + h) = hg'(x) + g(x)$ by cross-multiplying

$$k'(x) = \lim_{h\to 0}\frac{(hf'(x) + f(x))(hg'(x) + g(x)) - f(x)g(x)}{h}$$

$$k'(x) = \lim_{h\to 0}\frac{h^2f'(x)g'(x) + hf'(x)g(x) + hf(x)g'(x) + f(x)g(x) - f(x)g(x)}{h}$$

$$k'(x) = \lim_{h\to 0}\frac{h[hf'(x)g'(x) + f'(x)g(x) + f(x)g'(x)]}{h}$$

$$k'(x) = \lim_{h\to 0} hf'(x)g'(x) + f'(x)g(x) + f(x)g'(x)$$

$$k'(x) = f'(x)g(x) + f(x)g'(x)$$

Therefore if $k(x) = f(x)g(x)$ then $k'(x) = f'(x)g(x) + f(x)g'(x)$

Using different notations for $y = uv$, where u and v are both functions of x then

$y' = u'v + uv'$ or with Liebnitz notation

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

## Example

Find the equation of the normal to the curve $y = x^3 - 3x^2 + 4x + 1$ at the point (2, 5).

**Solution**

We begin by calculating the gradient of the curve by differentiating and setting $x = 2$.

$$\frac{dy}{dx} = 3x^2 - 6x + 4$$

At $x = 2$ $\quad \frac{dy}{dx} = 4$

Gradient of the normal is the negative reciprocal of 4, which is $\frac{^-1}{4}$

The normal is $y - 5 = \frac{^-1}{4}(x - 2)$

which simplifies to $x + 4y - 22 = 0$

## Example

If $y = \ln(kx - 1) - x^2$, and the gradient of the tangent to the curve at $x = 2$ is 1, find the value of k.

**Solution**

We begin by calculating the gradient of the curve by differentiating and setting $x = 2$.

$$\frac{dy}{dx} = \frac{k}{kx-1} - 2x$$

At $x = 2$, $\quad \frac{k}{2k-1} - 4 = 1$

Solving for k, $\quad \frac{k}{2k-1} = 5$

$$10k - 5 = k$$

$$9k = 5$$

$$k = \frac{5}{9} \ (0.5556)$$

**Achievement** – Answer the following questions.

**244.** Find the coordinates of the point on the curve $y = x^2 - \ln x$ where the gradient is 1.

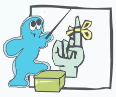
**245.** Find the x values of the points on the curve $y = \frac{3}{x} + \frac{x}{3}$ where the gradient equals $^-1$.

**246.** Find the gradient of the normal to the curve $y = \frac{1}{(x+1)}$, where $x = {}^-3$.

**247.** Find the equation of the normal to the curve $y = x^2 - 4x$ at $x = {}^-1$.

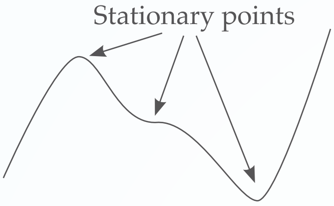
**248.** Find the equation of the tangent to the curve $y = 3.68\,e^{0.5x}$ at (2, 10).

**249.** Find the equation of the normal to the curve $y = 3.68\,e^{0.5x}$ at $x = 1$.

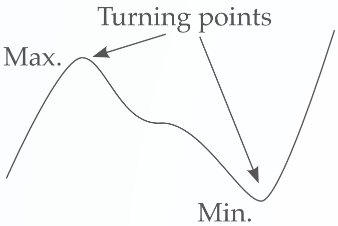


**Stationary points, turning points and maximum and minimum points, increasing and decreasing.**

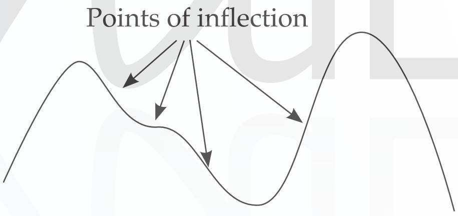
**A stationary point gets its name from the curve being momentarily stationary (not increasing or decreasing). Momentarily the gradient is zero.**




**The maximum and minimum points are called turning points because at these points the curve turns around and heads the other way. All turning points are stationary points.**



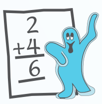

**Points of inflection are where the concavity of a curve changes. They can be stationary points but not necessarily. Points of inflection are never turning points.**




**To identify whether a function is increasing or decreasing we can use the derivative.**

**If $f'(x) > 0$ at each point in an interval, then the function is said to be increasing on that interval.**

**Similarly if $f'(x) < 0$ at each point in an interval then the function is said to be decreasing on that interval.**




## Example

Find the coordinates of all the stationary points (maximum and minimum) of the function

$$f(x) = x^3 + x^2 - x + 1$$

and state their nature (what type they are).

Identify when the function f(x) is decreasing and increasing.


To find the stationary points we begin by calculating the derivative

$$f(x) = x^3 + x^2 - x + 1$$

$$f'(x) = 3x^2 + 2x - 1$$

Setting $f'(x) = 0$

to identify the points which have a gradient of 0.

$$3x^2 + 2x - 1 = 0$$

$$(3x - 1)(x + 1) = 0 \qquad \text{factorising}$$

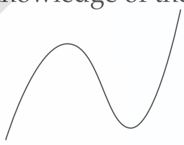
gives $x = {}^-1$ or $x = \frac{1}{3}$

when $x = {}^-1$ $\quad f({}^-1) = 2$ $\quad$ y coordinate

when $x = \frac{1}{3}$ $\quad f\left(\frac{1}{3}\right) = \frac{22}{27}$ $\quad$ y coordinate

Stationary coordinates are $({}^-1, 2)$ and $\left(\frac{1}{3}, \frac{22}{27}\right)$.

To identify the nature of the stationary points we could use our knowledge of the shape of a positive cubic.



Therefore the first x value ($x = {}^-1$) is going to be the maximum and the second value ($x = \frac{1}{3}$) will be the minimum.

Alternatively we could calculate the gradient before, between and after the turning points. We only need the sign of the derivative or gradient function. For a value before ${}^-1$ we have used ${}^-2$, for between the ${}^-1$ and $\frac{1}{3}$ we have used 0 and for after $\frac{1}{3}$ we have used 2. We substitute these values into $f'(x) = 3x^2 + 2x - 1$.

| x | ⁻2 | ⁻1 | 0 | $\frac{1}{3}$ | 2 |
|---|---|---|---|---|---|
| f′(x) | 7 | 0 | ⁻1 | 0 | 15 |
| Grad. | / | – | \ | – | / |

so the maximum point (∩) is $({}^-1, 2)$ and the minimum point (∪) is $\left(\frac{1}{3}, \frac{22}{27}\right)$.

The function $f(x) = x^3 + x^2 - x + 1$ is decreasing in the interval ${}^-1 < x < \frac{1}{3}$.

The function $f(x) = x^3 + x^2 - x + 1$ is increasing when $x < {}^-1$ and $x > \frac{1}{3}$.

## Example

Two children are making a large spherical snowball. If the volume is increasing at $0.75\ m^3/min$ when the radius is 0.85 m, find the rate the radius is increasing.

The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

**Solution**

❶ The rate of change of volume gives us

$$\frac{dV}{dt} = 0.75\ m^3/min$$

❷ We are required to find $\frac{dr}{dt}$.

❸ Reference to a sphere (with or without the formula) gives us

$$V = \frac{4}{3}\pi r^3$$

We write out the chain rule starting with what we require

$$\frac{dr}{dt} = \frac{dr}{d?} \times \frac{d?}{dt}$$

where the question mark could refer to any variable. Check the information given in the question. In this case it is V for volume

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

we have been given $\frac{dV}{dt} = 0.75$

and we work out $\frac{dr}{dV}$ from the formula

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2 \quad \text{differentiating}$$

when r = 0.85

$$\frac{dV}{dr} = 4\pi(0.85)^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi(0.85)^2}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi(0.85)^2} \cdot 0.75$$

$$= 0.083\ m/min \quad (2\ sf)$$

or $\frac{dr}{dt} = 8.3\ cm/min. \quad (2\ sf)$

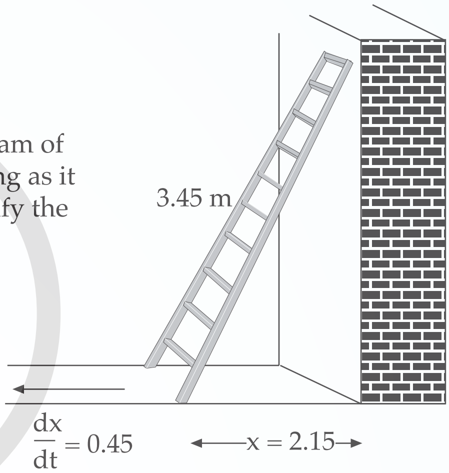
## Example

A ladder 3.45 m long is leaning against a wall.

The base of the ladder starts slipping at 0.45 m/s. Find the rate the top is sliding down the wall when the base is 2.15 m from the wall.

**Solution**

We draw a diagram of what is happening as it is easier to identify the three parts.




❶ We are given $\frac{dx}{dt} = 0.45$

❷ We require $\frac{dh}{dt}$.

❸ Using Pythagoras we have a relationship between h and x.

$$x^2 + h^2 = 3.45^2$$

$$h^2 = 3.45^2 - x^2$$

when x = 2.15, h = 2.70 m

We write out the chain rule starting with $\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{dh}{d?} \times \frac{d?}{dt}$$

$$= \frac{dh}{dx} \times \frac{dx}{dt}$$

To get $\frac{dh}{dx}$ we differentiate implicitly

$$h^2 = 3.45^2 - x^2$$

$$2h\frac{dh}{dx} = {}^-2x \qquad \text{but } x = 2.15,\ h = 2.7$$

$$5.4\frac{dh}{dx} = {}^-4.3$$

$$\frac{dh}{dx} = {}^-0.7963$$

Now substitute both $\frac{dh}{dx}$ and $\frac{dx}{dt}$ in the chain rule

$$\frac{dh}{dt} = {}^-0.7963 \times 0.45$$

$$= {}^-0.36\ m/s \quad (2\ dp)$$

i.e. $\frac{dh}{dt} = 36\ cm/s$ down.

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**186.** $\frac{dy}{dx} = 2x\cos x - x^2\sin x$

**187.** $\frac{dy}{dx} = 3\tan 3x + (9x - 6)\sec^2 3x$

**188.** $\frac{dy}{dx} = 2x\cot(5x - 1) - 5(x^2 - 1)\operatorname{cosec}^2(5x - 1)$

**189.** $\frac{dy}{dx} = 3x^2\operatorname{cosec} 2x - 2x^3\operatorname{cosec} 2x\cot 2x$

**190.** $h'(x) = 4\cos^2 x - 4\sin^2 x$

**191.** $f'(x) = 24\sin x\cos^2 x - 12\sin^3 x$

**192.** $h'(x) = \sec x\tan^2 x + \sec^3 x$

**193.** $k'(x) = \sin x(\sec^2 x + 1)$

**194.** $f'(x) = {}^{-}24\sin 4x\cos 4x$

**195.** $q'(x) = 3x^4\cos x + 12x^3\sin x$

**196.** $q'(x) = (12x^3 + 6x^2 + 4x + 1)e^{3x^2+2}$

**197.** $q'(x) = 6x\ln(3x-1) + \frac{9x^2}{3x-1}$

**198.** $\frac{dy}{dx} = (a + 2a^2x^2 + 2abx)e^{ax^2+b}$

**199.** $f'(x) = ae^{ax+b}\ln(ax+b) + \frac{ae^{ax+b}}{ax+b}$

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**200.** $g'(x) = \frac{6x^2 - 6x - 1}{(2x-1)^2}$

**201.** $g'(x) = \frac{30x^2 - 16x - 17}{(2x^2 + 8x - 1)^2}$

**202.** $\frac{dy}{dx} = \frac{2x - 3x^2}{2\sqrt{x}(x^2 - 2x)^2}$

**203.** $\frac{dy}{dx} = \frac{{}^{-}(4x-5)}{2\sqrt{x}(4x+5)^2}$

**204.** $g'(x) = \frac{12x + 1 - 15x^2}{3x^{2/3}(3x^2 - 6x + 1)^2}$

**205.** $h'(x) = \frac{x^{1/3} + 3}{6\sqrt{x}(x^{1/3} + 1)^2}$

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**206.** $\frac{dy}{dx} = \frac{60x^2 + 64x + 8}{e^{3x}(5x^2 + 2x)^2}$

**207.** $q'(x) = \frac{8x^2 + 8x - 2}{e^x(4x^2 - 1)^{3/2}}$

**208.** $\frac{dy}{dx} = \frac{(24x^3 - 18x)e^{x^2+1}}{(4x^2 - 1)^{3/2}}$

**209.** $f'(x) = \frac{(2x^3 - 8x)e^{x^2}}{(x^2 - 3)^2}$

**Page 42 cont...**

**210.** $\frac{dy}{dx} = \frac{(48x^2 - 12x - 3)e^{4x}}{x^2(4x^2 + 1)^2}$

**211.** $\frac{dy}{dx} = \frac{1 - 2\ln 2x}{x^3}$

**212.** $\frac{dy}{dx} = \frac{2ae^{ax}(ax^2 + b - 2x)}{(ax^2 + b)^2}$

**213.** $\frac{dy}{dx} = \frac{(x-b) - 2x\ln(x+b)}{(x^2 - b^2)^2}$

**214.** $\frac{dy}{dx} = \frac{{}^{-}1}{1 + \sin x}$

**215.** $f'(x) = 8x\cos 4x - 4(4x^2 - 1)\sin 4x$

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**216.** $m'(t) = \frac{2\cos 2t(1 - t^2) + 2t\sin 2t}{(1 - t^2)^2}$

**217.** $f'(x) = \frac{a(1 + \sin x - x\cos x)}{(1 + \sin x)^2}$

**Page 44**

**218.** $f'(x) = \frac{{}^{-}8}{x^3} - 10x$

**219.** $g'(x) = 15x^2 + 7 - \frac{3}{x^2}$

**220.** $\frac{dy}{dx} = 4x^3$

**221.** $h'(r) = 4\pi r + \frac{3}{\pi}$

**222.** $\frac{dy}{dx} = {}^{-}10e^{2x}(1 - e^{2x})^4$

**223.** $\frac{dy}{dx} = \frac{1}{x} + 2e^{2x}$

**224.** $m'(x) = \frac{\sec^2 x}{2\sqrt{\tan x}}$

**225.** $p'(x) = \frac{-11}{(2x-3)^2}$

**226.** $k'(x) = \frac{2x}{x^2 + 2}$

**227.** $\frac{dy}{dx} = \frac{\sec^2 x}{\tan x} = \operatorname{cosec} x\sec x$

**228.** $f'(x) = \frac{6}{5}(3x+5)^{-3/5}$

**229.** $\frac{dy}{dx} = \frac{{}^{-}4}{(3x-1)^2}$

**230.** $k'(x) = \frac{{}^{-}8}{x^3} + \frac{2}{x^2} - \frac{9}{x^4}$

**231.** $\frac{dy}{dx} = 4(6x - 2)(3x^2 - 2x + 1)^3$

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**232.** $f'(x) = (10x^2 + 5)e^{x^2}$

**233.** $g'(x) = \frac{e^{2x}}{2\sqrt{x}} + 2\sqrt{x}e^{2x}$

$g'(x) = \frac{e^{2x}(1 + 4x)}{2\sqrt{x}}$

**234.** $\frac{dy}{dx} = 10(2x^2 + x - 3)(4x^2 + 2x - 1)$

**235.** $g'(x) = 3e^{3x}\sin x + e^{3x}\cos x$

$g'(x) = e^{3x}(3\sin x + \cos x)$

**236.** $h'(x) = 3e^{3x}\ln(2x) + \frac{e^{3x}}{x}$

**237.** $j'(x) = \frac{(x^2 + 6)\sec x\tan x - 2x\sec x}{(x^2 + 6)^2}$

**238.** $k'(x) = \frac{12x^2(1 + 2\ln x) - 8x^2}{(1 + 2\ln x)^2}$

$k'(x) = \frac{4x^2(1 + 6\ln x)}{(1 + 2\ln x)^2}$

**239.** $\frac{dy}{dx} = \frac{2(1 - x^2)\cos x + 4x\sin x}{(1 - x^2)^2}$

**240.** $\frac{dy}{dx} = \frac{2\cos^2 x + 2\sin x + 2\sin^2 x}{\cos^2 x}$

$\frac{dy}{dx} = \frac{2}{1 - \sin x}$

or $\frac{dy}{dx} = 2 + 2\sec x\tan x + 2\tan^2 x$ if using the product rule.

**241.** $h(x)' = \frac{{}^{-}(\cos x + 1)}{\sin^2 x}$

**242.** $g'(x) = \frac{2(2x-1)^{-1/2}e^{3x} - 6e^{3x}(2x-1)^{1/2}}{(2e^{3x})^2}$

$g'(x) = \frac{(2 - 3x)e^{-3x}}{\sqrt{2x-1}}$

**243.** $k'(x) = 12(3x+1)^3(x-2)^{1/2} + \frac{1}{2}(3x+1)^4(x-2)^{-1/2}$

$k'(x) = \frac{(3x+1)^3(27x - 47)}{2(x-2)^{0.5}}$